8-2-5 角運動量IIの極座標表現

\[
\begin{align*}
\text{極座標} & \quad \begin{cases}
x = r \sin \theta \cos \phi \\
y = r \sin \theta \sin \phi \\
z = r \cos \theta
\end{cases} \\
2, \omega \\
\frac{\partial}{\partial r} & = \sin \theta \cos \phi \frac{\partial}{\partial x} + \sin \theta \sin \phi \frac{\partial}{\partial y} + \cos \theta \frac{\partial}{\partial z} \\
\frac{\partial}{\partial \theta} & = r \cos \theta \cos \phi \frac{\partial}{\partial x} + r \cos \theta \sin \phi \frac{\partial}{\partial y} - r \sin \theta \frac{\partial}{\partial z} \\
\frac{\partial}{\partial \phi} & = -r \sin \theta \sin \phi \frac{\partial}{\partial x} + r \sin \theta \cos \phi \frac{\partial}{\partial y}
\end{align*}
\]

[書く直す]
\[
R = \begin{pmatrix}
\sin \theta & \sin \phi & \cos \theta \\
\cos \theta & \cos \phi & -\sin \theta \\
-\sin \phi & \cos \phi & 0
\end{pmatrix}
\]

\[R^t = \begin{pmatrix}
\sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\
\sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\
\cos \theta & -\sin \theta & 0
\end{pmatrix}
\]

\[R^t \left( \begin{pmatrix}
\frac{\partial }{\partial r} \\
\frac{1}{r} \frac{\partial }{\partial \theta} \\
\frac{1}{r \sin \theta} \frac{\partial }{\partial \phi}
\end{pmatrix}
\right) = R^t \left( \begin{pmatrix}
\frac{\partial }{\partial x} \\
\frac{\partial }{\partial y} \\
\frac{\partial }{\partial z}
\end{pmatrix}
\right) = \left( \begin{pmatrix}
\frac{\partial }{\partial x} \\
\frac{\partial }{\partial y} \\
\frac{\partial }{\partial z}
\end{pmatrix}
\right)
\]

\[
\begin{cases}
\frac{\partial }{\partial x} = \sin \theta \cos \phi \frac{\partial }{\partial r} + \frac{\cos \theta \cos \phi}{r} \frac{\partial }{\partial \theta} - \frac{\sin \phi}{r \sin \theta} \frac{\partial }{\partial \phi} \\
\frac{\partial }{\partial y} = \sin \theta \sin \phi \frac{\partial }{\partial r} + \frac{\cos \theta \sin \phi}{r} \frac{\partial }{\partial \theta} + \frac{\cos \phi}{r \sin \theta} \frac{\partial }{\partial \phi} \\
\frac{\partial }{\partial z} = \cos \theta \frac{\partial }{\partial r} - \frac{\sin \theta}{r} \frac{\partial }{\partial \theta}
\end{cases}
\]
この関係式を用いると、L の極座標表示は次のようになる。

[例]

\[ L_z = -i \hbar \left( r \frac{\partial}{\partial y} - y \frac{\partial}{\partial r} \right) = -i \hbar \left[ r \sin\theta \cos\phi \left( \sin\theta \sin\phi \frac{\partial}{\partial r} \right. \right. \]
\[ + \frac{\cos\theta \sin\phi}{r} \frac{\partial}{\partial \theta} + \frac{\cos\phi}{r \sin\theta} \frac{\partial}{\partial \phi} \right) \]
\[ \left. \left. - r \sin\theta \sin\phi \left( \sin\theta \cos\phi \frac{\partial}{\partial r} + \frac{\cos\theta \cos\phi}{r} \frac{\partial}{\partial \theta} - \frac{\sin\phi}{r \sin\theta} \frac{\partial}{\partial \phi} \right) \right] \right] \]
\[ = -i \hbar \frac{\partial}{\partial \phi} \]

\[ \therefore \quad L_z = -i \hbar \frac{\partial}{\partial \phi} \]

同様に \( \frac{\partial \phi}{\partial r} \) についても

\[ L_x = -i \hbar \left[ -\sin\phi \frac{\partial}{\partial \theta} - \cos\theta \cos\phi \frac{\partial}{\partial \phi} \right] \]
\[ L_y = -i \hbar \left[ \cos\phi \frac{\partial}{\partial \theta} - \cos\theta \sin\phi \frac{\partial}{\partial \phi} \right] \]

か \( \frac{\partial \phi}{\partial r} \) の \( L_z \)